Problem 43)

a) The change of variable $y = x^3$ yields

$$\mathrm{d}y = 3x^2\mathrm{d}x = 3y^{2/3}\mathrm{d}x \;\rightarrow\; \mathrm{d}x = \tfrac{1}{3}y^{-2/3}\mathrm{d}y. \tag{1}$$

Consequently,

$$\int_0^\infty \exp(-x^3)\,\mathrm{d}x = \tfrac{1}{3}\int_0^\infty y^{(1/3\,-\,1)}\exp(-y)\,\mathrm{d}y = \tfrac{1}{3}\Gamma(\tfrac{1}{3}). \tag{2}$$

b) The Taylor series exapnsion of $g(x)$ around the stationary point x_0 is written as

$$g(x) = g(x_0) + \cancel{g'(x_0)}(x - x_0) + \tfrac{1}{2!}\cancel{g''(x_0)}(x - x_0)^2 + \tfrac{1}{3!}g'''(x_0)(x - x_0)^3 + \cdots. \tag{3}$$

We thus have

$$\int_a^b f(x)\exp[\mathrm{i}\eta g(x)]\,\mathrm{d}x \cong f(x_0)e^{\mathrm{i}\eta g(x_0)}\int_a^b \exp[\mathrm{i}(\eta/6)g'''(x_0)(x - x_0)^3]\,\mathrm{d}x$$

Extend the domain of integration from (a, b) to $(-\infty, \infty)$. → Change of variable: $y = x - x_0$ →

$$\cong f(x_0)e^{\mathrm{i}\eta g(x_0)}\int_{-\infty}^{\infty} e^{\mathrm{i}(\eta/6)g'''(x_0)y^3}\,\mathrm{d}y$$

$$= f(x_0)e^{\mathrm{i}\eta g(x_0)}\Big[\int_{-\infty}^{0} e^{\mathrm{i}(\eta/6)g'''(x_0)y^3}\mathrm{d}y + \int_0^\infty e^{\mathrm{i}(\eta/6)g'''(x_0)y^3}\mathrm{d}y\Big]$$

Change of variable: $y \rightarrow -y$ →

$$= f(x_0)e^{\mathrm{i}\eta g(x_0)}\Big[\int_0^\infty e^{-\mathrm{i}(\eta/6)g'''(x_0)y^3}\mathrm{d}y + \int_0^\infty e^{\mathrm{i}(\eta/6)g'''(x_0)y^3}\mathrm{d}y\Big]$$

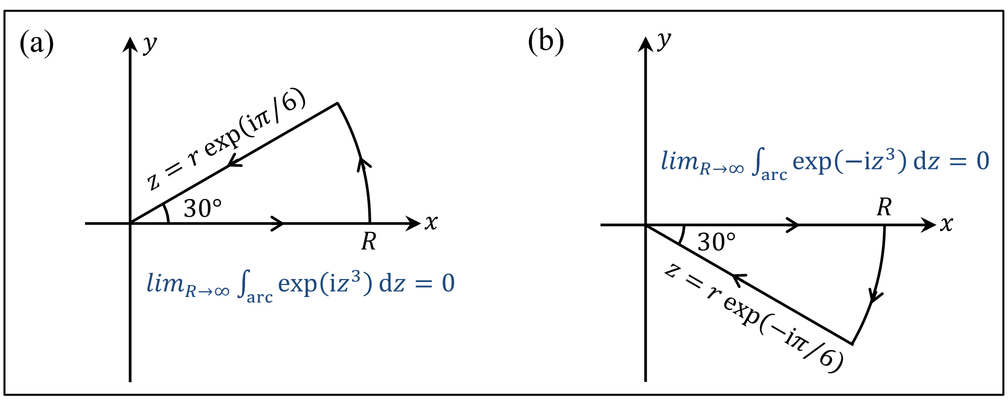
$$= 2f(x_0)e^{\mathrm{i}\eta g(x_0)}\mathrm{Re}\Big[\int_0^\infty e^{\mathrm{i}(\eta/6)g'''(x_0)y^3}\mathrm{d}y\Big]$$

Change of variable: $x = \sqrt[3]{|\eta g'''(x_0)/6|}\,y$ →

$$= \frac{2f(x_0)\exp[\mathrm{i}\eta g(x_0)]}{\sqrt[3]{|\eta g'''(x_0)/6|}}\mathrm{Re}\Big[\int_0^\infty \exp(\pm\mathrm{i}x^3)\,\mathrm{d}x\Big]. \tag{4}$$

← $\pm$ is the sign of $\eta g'''(x_0)$

The integral $\int_0^\infty \exp(\mathrm{i}x^3)\,\mathrm{d}x$ appearing on the last line of Eq.(4) may now be evaluated with the aid of the contour shown in Fig.(a) below.



$$\int_0^\infty \exp(\mathrm{i}x^3)\,\mathrm{d}x = \int_{30^\circ\,\mathrm{line}}\exp(\mathrm{i}z^3)\,\mathrm{d}z = \int_0^\infty \exp[\mathrm{i}(re^{\mathrm{i}\pi/6})^3]e^{\mathrm{i}\pi/6}\mathrm{d}r$$

$$= e^{\mathrm{i}\pi/6}\int_0^\infty \exp(-r^3)\mathrm{d}r = \tfrac{1}{3}\Gamma(\tfrac{1}{3})[\cos(\pi/6) + \mathrm{i}\sin(\pi/6)]. \tag{5}$$

The other integral, $\int_0^\infty \exp(-\mathrm{i}x^3)\,\mathrm{d}x$, is just the conjugate of the integral in Eq.(5). To evaluate this integral in the complex plane, one must use the contour shown in Fig.(b) to ensure that, in the limit when $R \rightarrow \infty$, the contribution of the arc vanishes. We will then have

$$\int_0^\infty \exp(-\mathrm{i}x^3)\,\mathrm{d}x = \int_{-30^\circ\,\mathrm{line}} \exp(-\mathrm{i}z^3)\,\mathrm{d}z = \int_0^\infty \exp[-\mathrm{i}(re^{-\mathrm{i}\pi/6})^3]e^{-\mathrm{i}\pi/6}\mathrm{d}r$$
$$= e^{-\mathrm{i}\pi/6}\int_0^\infty \exp(-r^3)\mathrm{d}r = ⅓\Gamma(⅓)[\cos(\pi/6) - \mathrm{i}\sin(\pi/6)]. \qquad (6)$$

Considering that $\cos(\pi/6) = \sqrt{3}/2$, our stationary-phase approximation finally yields

$$\int_a^b f(x)\exp[\mathrm{i}\eta g(x)]\,\mathrm{d}x \cong \frac{\Gamma(⅓)f(x_0)\exp[\mathrm{i}\eta g(x_0)]}{\sqrt{3}\sqrt[3]{|\eta g'''(x_0)/6|}}. \qquad (7)$$
